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ADDENDUM

Higher-order JWKB approximations for radial problems: III. The r^{2m} oscillator

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Abstract. Higher-order JWKB approximations are applied to the calculation of energy levels of an oscillator with the potential $V(r) = r^{2m}$ (m integer). The JWKB quantisation condition for the energy W is shown to be expressible as

$$(n+\tfrac{3}{2})\pi = AX + BX^{-1} + CX^{-3} + \ldots$$

where $X = W^{(m+1)/2m}$. The l-dependent coefficients A, B, C are determined exactly by taking into account contributions from all orders. On inversion the above series yields an explicit analytical formula for the energy levels. Extension of the result to d dimensions is immediate.

1. Introduction

Recently we have considered the application of higher-order JWKB approximations to the radial Schrödinger equation (Seetharaman and Vasan 1984, Vasan and Seetharaman 1984, to be referred to as I and II respectively). We derived in II an explicit analytical formula for the energy eigenvalues of the quartic oscillator $V(r) = r^4$, by including corrections up to the fourth order of the approximation. This formula was found to be in excellent agreement with known numerical eigenvalues. In this addendum we generalise the formula to the case of the oscillator with the potential $V(r) = r^{2m}$, m being a positive integer. First we show that the quantisation condition for the energy W can be put in the form

$$\pi(n+\tfrac{3}{2}) = AX + BX^{-1} + CX^{-3} + \ldots \tag{1}$$

where $X = W^{(m+1)/2m}$. We then show how the coefficients A, B and C can be evaluated exactly, after taking into account all orders of the approximation. An explicit formula for the energy is obtained by inverting the above series. As the formalism has been dealt with at length in II for the quartic oscillator, we shall present here only the essential details.

2. Energy quantisation condition

For the potential $V(r) = r^{2m}$, it proves convenient to define

$$z = r^2 W^{-1/m}, \qquad X = W^{(m+1)/2m}, \qquad \sigma = (l+\tfrac{1}{2})^2/2X^2, \qquad F(z) = z - z^{m+1} - \sigma. \tag{2}$$

The energy quantisation condition in the fourth order of the approximation can be written as

$$(2n_r+1)\pi = J_0+J_2+J_4 \tag{3}$$

where

$$J_0=(1/\sqrt{2})X\oint \mathrm{d}z\, z^{-1}F^{1/2},$$

$$J_2=\frac{m(m+1)}{24\sqrt{2}}X^{-1}\oint \mathrm{d}z\, z^{m}F^{-3/2}, \tag{4}$$

$$J_4=-\frac{\sqrt{2}\,m(m+1)}{128}X^{-3}\oint \mathrm{d}z[\tfrac{7}{6}m(m+1)z^{2m+1}F^{-7/2}+\tfrac{1}{15}(5m+6)(m-1)z^{m}F^{-5/2}].$$

The above expressions for the higher-order JWKB terms J_i have been obtained by putting $V(r)=r^{2m}$ in the formulae given in the appendix of II†. The contour of integration is traversed clockwise around the branch cut joining the two branch points of $F^{1/2}$ which are located at the classical turning points of the problem. To first order in σ, these turning points are at $z=\sigma$ and $z=1-\sigma/m$. The branch of $F^{1/2}$ chosen is that which is positive real on the upper lip of the cut.

2.1. σ expansion of the integrals

It is clear that in order to cast the quantisation condition into the form (1), we should expand the integrals in J_i and retain all terms up to $O(\sigma^2)$ in J_0, $O(\sigma)$ in J_2 and $O(1)$ in J_4. This expansion is valid, since all our contour integrals have a well defined small σ expansion.

Let $I_i(\sigma)$ denote the integral alone in J_i. Considering first I_0, one finds after a careful inspection that its σ expansion is of the form

$$I_0(\sigma)=a_0+b_0\sqrt{\sigma}+c_0\sigma+d_0\sigma^{3/2}+e_0\sigma^2+\dots. \tag{5}$$

The coefficients in (5) can be determined successively by differentiating I_0 and setting $\sigma=0$. In this process, we must evaluate integrals of the form

$$\oint \mathrm{d}z\, z^{a/2}(1-z^m)^{b/2}$$

with the branch points now at $z=0$ and $z=1$. The evaluation of such integrals is outlined in the appendix. Using the result (A1) of the appendix, we find

$$a_0=Z(-1,1), \qquad b_0=-2\pi, \qquad c_0=-mZ(2m-3,3)/2, \qquad d_0=0,$$
$$e_0=-mZ(2m-5,-5)/8. \tag{6}$$

We take I_2 next. Its expansion is

$$I_2(\sigma)=a_2+b_2\sigma+\dots. \tag{7}$$

Evaluating the coefficients as before, we get

$$a_2=Z(2m-3,-3), \qquad b_2=3Z(2m-5,-5)/2. \tag{8}$$

† The formula for J_4 given in the appendix of II should read … $16r^{-3}$ … not … $16r$….

Finally, the value of I_4 $(\sigma = 0) = a_4$ is found to be

$$a_4 = 7m(m+1)Z(4m-5, -7)/2 + (5m+6)(m-1)Z(2m-5, -5)/15. \tag{9}$$

Adding all these terms and grouping like powers of X, we find

$$\begin{aligned} J_0 + J_2 + J_4 &= \frac{1}{\sqrt{2}} a_0 X - \pi X\sqrt{2\sigma} + \frac{1}{\sqrt{2}}\left(c_0 \sigma X + \frac{m(m+1)}{24} a_2 X^{-1} \right) \\ &\quad + \frac{1}{\sqrt{2}}\left(e_0 \sigma^2 X + \frac{m(m+1)}{24} b_2 \sigma X^{-1} - \frac{m(m+1)}{64} a_4 X^{-3} \right) + \dots \\ &= AX - \pi(l+\tfrac{1}{2}) + BX^{-1} + CX^{-3} + \dots \end{aligned} \tag{10}$$

where the coefficients A, B, C have the following expressions:

$$\begin{aligned} A &= \sqrt{2}\, B(\tfrac{1}{2}, 1/2m)/(m+1), \\ B &= (m-1)[-(m+1)/6 + (l+\tfrac{1}{2})^2] B(\tfrac{1}{2}, 1-1/2m)/\sqrt{2}\, 2m, \\ C &= (9-m^2)[(m+1)(m-3)(4m+3)/60 + (m+1)(l+\tfrac{1}{2})^2 - (l+\tfrac{1}{2})^4] \\ &\qquad \times B(\tfrac{1}{2}, 1-3/2m)/48\sqrt{2}\, m^2, \end{aligned} \tag{11}$$

where we have used the explicit values of $Z(a, b)$ in terms of beta functions (see (A1)) and used the properties of the beta function to rearrange the arguments. We may note that the leading term of the sixth order of the approximation is proportional to X^{-5}. Therefore, inclusion of terms of order higher than J_4 in (3) will not change the values of A, B, C given in (11). In this sense, these coefficients are exact.

2.2. *Analytical formula for W*

The quantisation condition (3) now takes the form

$$\pi(n+\tfrac{3}{2}) = AX + BX^{-1} + CX^{-3} + O(X^{-5}) \tag{12}$$

where $n = 2n_r + l$. A formula for W can be obtained by inverting (12). The result is

$$W = aN^{2m/(m+1)}(1 + bN^{-2} + cN^{-4} + \dots) \tag{13}$$

with $N = n + \frac{3}{2}$. The coefficients a, b, c are related to A, B, C as follows:

$$\begin{aligned} &a = (A/\pi)^{-2m/(m+1)}, \qquad b = -2mAB/(m+1)\pi^2, \\ &c = -[mA^2/(m+1)^2\pi^4][(m+3)B^2 + 2(m+1)CA]. \end{aligned} \tag{14}$$

The above relations provide an explicit analytical expression for the energy levels. We may note that b and c are l dependent, while a is not. The energy levels of the r^{2m} oscillator in d dimensions may be readily obtained from our formula by replacing l by $l + \frac{1}{2}(d-3)$, and $n + \frac{3}{2}$ by $n + d/2$.

3. Discussion

The general formula for the energy levels of the r^{2m} oscillator derived above gives the correct harmonic oscillator spectrum for $m = 1$, and, for $m = 2$, it is identical to the formula for the quartic oscillator given in II. As noted there, our formula is very accurate for the quartic oscillator in two and three dimensions for which exact (1 in 10^{15}) numerical eigenvalues are available. When $l = 0$, the coefficients A, B, C become identical to the corresponding ones for the x^{2m} oscillator computed by Bender *et al* (1977). This reveals that the JWKB approximation preserves the following correspondence between the exact eigenvalues of the three-dimensional Schrödinger equation and its one-dimensional counterpart: the s-wave energy levels in the potential $V(r)$ should coincide with the odd parity levels in the one-dimensional potential $V(x)$, if $V(x)$ is taken to be symmetric. To our knowledge, a general formula such as the one above has not been reported in the literature.

It may be noted that the method outlined in this paper cannot be applied to the potentials $V(r) = r^{2m+1}$. For these potentials, the contour of integration gets pinched between two singularities when $\sigma = 0$. Consequently, it is not possible to expand the integrals in powers of σ.

Appendix

The integrals to be evaluated are all of the form

$$Z(a, b) = \oint dz\, z^{a/2}(1 - z^m)^{b/2}$$

where a, b are odd integers. The contour surrounds the branch cut joining $z = 0$ and $z = 1$. As the integral is well defined, we can replace $1 - z^m$ by $\lambda - z^m$ and finally set $\lambda = 1$. In this way, we can express Z as the derivative with respect to λ of another integral in which b has the value $+1$ or -1. Then, by a suitable change of variable, the λ dependence of the integral can be factored out and the $\lambda \to 1$ limit can be evaluated. The remaining integral, which now has only integrable singularities at $z = 0$ and $z = 1$, can be done by compressing the contour until it coincides with the lips of the cut. The following calculation illustrates the procedure.

$$
\begin{aligned}
Z(2m-5,&-5)\\
&= \oint dz\, z^{m-5/2}(1 - z^m)^{-5/2}\\
&= \lim_{\lambda \to 1} \frac{4}{3} \frac{\partial^2}{\partial \lambda^2} \oint dz\, z^{m-5/2}(\lambda - z^m)^{-1/2}\\
&= \lim \frac{4}{3m} \frac{\partial^2}{\partial \lambda^2} \lambda^{(m-3)/2m} \oint dy\, y^{-3/2m}(1 - y)^{-1/2}, \qquad \lambda y = z^m,\\
&= \frac{(9 - m^2)}{3m^3} \oint dy\, y^{-3/2m}(1 - y)^{-1/2}\\
&= \frac{9 - m^2}{3m^3} 2 \int_0^1 dx\, x^{-3/2m}(1 - x)^{-1/2}\\
&= \frac{2(9 - m^2)}{3m^2} B(\tfrac{1}{2}, 1 - 3/2m)
\end{aligned}
$$

where B is the beta function defined by $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$. In a similar manner, the other integrals can be evaluated. The general result is ·

$$Z(a, b) = (2/m)B(1+b/2, (a+2)/2m). \tag{A1}$$

For a given b, the first argument of B in (A1) can always be made $\frac{1}{2}$ by using the properties of the beta function.

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